

FURTHER REFINEMENTS OF THE CAUCHY–SCHWARZ INEQUALITY FOR MATRICES

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ABSTRACT. Let A, B and X be $n \times n$ matrices such that A, B are positive semidefinite. We present some refinements of the matrix Cauchy-Schwarz inequality by using some integration techniques and various refinements of the Hermite–Hadamard inequality. In particular, we establish the inequality

$$\begin{aligned} ||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 &\leq ||| |A^tXB^{1-s}|^r ||| \quad ||| |A^{1-t}XB^s|^r ||| \\ &\leq \max\{||| |AX|^r ||| \quad ||| |XB|^r |||, ||| |AXB|^r ||| \quad ||| |X|^r |||\}, \end{aligned}$$

where $s, t \in [0, 1]$ and $r \geq 0$.

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{M}_n be the C^* -algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathcal{M}_n$, we write $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$. We use \mathcal{S}_n for the set of positive semidefinite matrices and \mathcal{P}_n for the set of positive definite matrices in \mathcal{M}_n . A norm $||| \cdot |||$ is called unitarily invariant norm if $||| UAV ||| = ||| A |||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. The numerical range of $A \in \mathcal{M}_n$ is $W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ and the numerical radius of A is defined by $\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathbb{C}^n, \|x\| = 1\}$. It is well-known [5] that $\omega(\cdot)$ is a weakly unitarily invariant norm on \mathcal{M}_n , that is $\omega(U^*AU) = \omega(A)$ for every unitary $U \in \mathcal{M}_n$. The Hadamard product (Schur product) of two matrices $A, B \in \mathcal{M}_n$ is the matrix $A \circ B$ whose (i, j) entry is $a_{ij}b_{ij}$ ($1 \leq i, j \leq n$). The Schur multiplier operator S_A on \mathcal{M}_n is defined by $S_A = A \circ X$ ($X \in \mathcal{M}_n$). The induced norm of S_A with respect to the spectral norm is $\|S_A\| = \sup_{X \neq 0} \frac{\|S_A(X)\|}{\|X\|} = \sup_{X \neq 0} \frac{\|A \circ X\|}{\|X\|}$, and the induced norm of S_A with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$

A continuous real valued function f on an interval $J \subseteq \mathbb{R}$ is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all $A, B \in \mathcal{M}_n$ with spectra in J . Recall that a real

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valued function F defined on $J_1 \times J_2$ is called convex if

$$F(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \leq \lambda F(x_1, y_1) + (1 - \lambda)F(x_2, y_2)$$

for all $x_1, x_2 \in J_1, y_1, y_2 \in J_2$ and $\lambda \in [0, 1]$.

For two sequences $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ of real numbers, the classical Cauchy-Schwarz inequality states that

$$\left(\sum_{j=1}^n a_j b_j \right)^2 \leq \left(\sum_{j=1}^n a_j^2 \right) \left(\sum_{j=1}^n b_j^2 \right)$$

with equality if and only if the sequences a and b are proportional [11]. Horn and Mathias [7] gave a matrix Cauchy-Schwarz inequality as follows

$$||| |A^* B|^r |||^2 \leq ||| (AA^*)^r ||| \quad ||| (BB^*)^r ||| \quad (A, B, X \in \mathcal{M}_n, r \geq 0).$$

Bhatia and Davis [2] showed that

$$||| |A^* X B|^r |||^2 \leq ||| |AA^* X|^r ||| \quad ||| |X B B^*|^r ||| \quad (A, B, X \in \mathcal{M}_n, r \geq 0), \quad (1.1)$$

which is equivalent to

$$||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 \leq ||| |A X|^r ||| \quad ||| |X B|^r ||| \quad (A, B \in \mathcal{S}_n, X \in \mathcal{M}_n, r \geq 0). \quad (1.2)$$

In [6] it is proved that the function $f(t) = ||| |A^t X B^{1-t}|^r ||| \quad ||| |A^{1-t} X B^t|^r |||$ is convex on the interval $[0, 1]$, when $A, B \in \mathcal{S}_n, X \in \mathcal{M}_n$ and attains its minimum at $t = \frac{1}{2}$. In view of the fact that the function f is decreasing on the interval $[0, \frac{1}{2}]$ and increasing on the interval $[\frac{1}{2}, 1]$. In particular, we have a refinement of the Cauchy-Schwarz inequality [6] as follows

$$||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 \leq ||| |A^\mu X B^{1-\mu}|^r ||| \quad ||| |A^{1-\mu} X B^\mu|^r ||| \leq ||| |A X|^r ||| \quad ||| |X B|^r |||, \quad (1.3)$$

where $A, B \in \mathcal{S}_n, X \in \mathcal{M}_n$ and $\mu \in [0, 1]$.

Applying the convexity of the function $f(t) = ||| |A^t X B^{1-t}|^r ||| \quad ||| |A^{1-t} X B^t|^r |||$ ($t \in [0, 1]$), we show some refinements of inequality (1.3). we also show the convexity of the function $f(s, t) = ||| |A^s X B^{1-t}|^r ||| \quad ||| |A^{1-s} X B^t|^r |||$ and present some other refinements of inequality (1.3). In the last section we show some related numerical radius inequalities.

2. NORM INEQUALITY INVOLVING THE CAUCHY-SCHWARZ

In this section, we establish some refinements of inequality (1.3). To this end, we need the following Hermite-Hadamard inequality.

Lemma 2.1. [4] *Let g be a real-valued convex function on $[a, b]$. Then*

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(s)ds \leq \frac{1}{4} \left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b) \right] \leq \frac{g(a) + g(b)}{2}.$$

Applying Lemma 2.1 we have following result.

Proposition 2.2. *Suppose that $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and $r \geq 0$. Then*

$$\begin{aligned} ||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 &\leq \frac{1}{|1-2\mu|} \left| \int_{\mu}^{1-\mu} ||| |A^sXB^{1-s}|^r ||| \quad ||| |A^{1-s}XB^s|^r ||| ds \right| \\ &\leq \frac{1}{2} \left[||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 + ||| |A^{\mu}XB^{1-\mu}|^r ||| \quad ||| |A^{1-\mu}XB^{\mu}|^r ||| \right] \\ &\leq ||| |A^{\mu}XB^{1-\mu}|^r ||| \quad ||| |A^{1-\mu}XB^{\mu}|^r ||| \end{aligned}$$

for all $0 \leq \mu \leq 1$ and all unitarily invariant norms $||| \cdot |||$.

Proof. Let $f(t) = ||| |A^tXB^{1-t}|^r ||| \quad ||| |A^{1-t}XB^t|^r |||$. First assume that $0 \leq \mu < \frac{1}{2}$. It follows from Lemma 2.1 that

$$\begin{aligned} f\left(\frac{\mu+1-\mu}{2}\right) &\leq \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} f(s)ds \\ &\leq \frac{1}{4} \left[f(\mu) + 2f\left(\frac{\mu+1-\mu}{2}\right) + f(1-\mu) \right] \\ &\leq \frac{f(1-\mu) + f(\mu)}{2}, \end{aligned}$$

whence

$$f\left(\frac{1}{2}\right) \leq \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} f(s)ds \leq \frac{1}{2} \left[f(\mu) + f\left(\frac{1}{2}\right) \right] \leq f(\mu).$$

Hence

$$\begin{aligned} ||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 &\leq \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} ||| |A^{1-s}XB^s|^r ||| \quad ||| |A^sXB^{1-s}|^r ||| ds \\ &\leq \frac{1}{2} \left[||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 + ||| |A^{\mu}XB^{1-\mu}|^r ||| \quad ||| |A^{1-\mu}XB^{\mu}|^r ||| \right] \\ &\leq ||| |A^{\mu}XB^{1-\mu}|^r ||| \quad ||| |A^{1-\mu}XB^{\mu}|^r |||. \end{aligned} \tag{2.1}$$

Now, assume that $\frac{1}{2} < \mu \leq 1$. By the symmetry property of (2.1) with respect to μ , if we replace μ by $1-\mu$, then

$$\begin{aligned} ||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 &\leq \frac{1}{2\mu-1} \int_{1-\mu}^{\mu} ||| |A^{1-s}XB^s|^r ||| \quad ||| |A^sXB^{1-s}|^r ||| ds \\ &\leq \frac{1}{2} \left[||| |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r |||^2 + ||| |A^{\mu}XB^{1-\mu}|^r ||| \quad ||| |A^{1-\mu}XB^{\mu}|^r ||| \right] \\ &\leq ||| |A^{\mu}XB^{1-\mu}|^r ||| \quad ||| |A^{1-\mu}XB^{\mu}|^r |||. \end{aligned} \tag{2.2}$$

Since $\lim_{\mu \rightarrow \frac{1}{2}} \frac{1}{|2\mu-1|} \left| \int_{\mu}^{1-\mu} ||| |A^s X B^{1-s}|^r ||| \quad ||| |A^{1-s} X B^s|^r ||| ds \right| = ||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2$, inequalities (2.1) and (2.2) yield the desired result. \square

Now, we show the convexity of the function

$$F(s, t) = ||| |A^{1-t} X B^{1+s}|^r ||| \quad ||| |A^{1+t} X B^{1-s}|^r |||$$

and we use the convexity of F to prove some Cauchy-Schwarz type inequalities.

Theorem 2.3. *Suppose that $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and $r \geq 0$. Then the function*

$$F(s, t) = ||| |A^{1-t} X B^{1+s}|^r ||| \quad ||| |A^{1+t} X B^{1-s}|^r |||$$

is convex on $[-1, 1] \times [-1, 1]$ and attains its minimum at $(0, 0)$.

Proof. The function F is continuous and $F(s, t) = F(-s, -t)$ ($s, t \in [0, 1]$). Thus it is enough to show that

$$F(s_1, t_1) \leq \frac{1}{2} [F(s_1 + s_2, t_1 + t_2) + F(s_1 - s_2, t_1 - t_2)],$$

where $s_1 \pm s_2, t_1 \pm t_2 \in [-1, 1] \times [-1, 1]$.

Let $s_1 \pm s_2, t_1 \pm t_2 \in [-1, 1] \times [-1, 1]$. Applying inequality (1.1) we obtain

$$\begin{aligned} ||| |A^{1-t_1} X B^{1+s_1}|^r ||| &= ||| |A^{t_2} (A^{1-t_1-t_2} X B^{1+s_1-s_2}) B^{s_2}|^r ||| \\ &\leq \left\{ ||| |A^{1-(t_1-t_2)} X B^{1+(s_1-s_2)}|^r ||| \quad ||| |A^{1-(t_1+t_2)} X B^{1+(s_1+s_2)}|^r ||| \right\}^{1/2} \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} ||| |A^{1+t_1} X B^{1-s_1}|^r ||| &= ||| |A^{t_2} (A^{1+t_1-t_2} X B^{1-s_1-s_2}) B^{s_2}|^r ||| \\ &\leq \left\{ ||| |A^{1+(t_1+t_2)} X B^{1-(s_1+s_2)}|^r ||| \quad ||| |A^{1+(t_1-t_2)} X B^{1-(s_1-s_2)}|^r ||| \right\}^{1/2}. \end{aligned} \quad (2.4)$$

Applying (2.3), (2.4) and the arithmetic-geometric mean inequality we get

$$\begin{aligned} F(s_1, t_1) &= ||| |A^{1-t_1} X B^{1+s_1}|^r ||| \quad ||| |A^{1+t_1} X B^{1-s_1}|^r ||| \\ &\leq [F(s_1 + s_2, t_1 + t_2) F(s_1 - s_2, t_1 - t_2)]^{1/2} \\ &\leq \frac{1}{2} [F(s_1 + s_2, t_1 + t_2) + F(s_1 - s_2, t_1 - t_2)]. \end{aligned}$$

\square

Corollary 2.4. *Suppose that $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and $r \geq 0$. Then*

$$\begin{aligned} ||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 &\leq ||| |A^t X B^{1-s}|^r ||| \quad ||| |A^{1-t} X B^s|^r ||| \\ &\leq \max\{ ||| |AX|^r ||| \quad ||| |XB|^r |||, ||| |AXB|^r ||| \quad ||| |X|^r ||| \}, \end{aligned}$$

where $s, t \in [0, 1]$.

Proof. If we replace s, t, A, B by $2s - 1, 2t - 1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively, in Theorem 2.3, we get the function $G(s, t) = ||| |A^t X B^{1-s}|^r ||| ||| |A^{1-t} X B^s|^r |||$ is convex on $[0, 1] \times [0, 1]$ and attains its minimum at $(\frac{1}{2}, \frac{1}{2})$. Hence

$$||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 \leq ||| |A^t X B^{1-s}|^r ||| ||| |A^{1-t} X B^s|^r |||.$$

In addition, since the function G is continuous and convex on $[0, 1] \times [0, 1]$, it follows that G attains its maximum at the vertices of the square. Moreover, due to the symmetry there are two possibilities for the maximum. \square

Dragomir [3, p. 316] proved that

$$\begin{aligned} F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b F(x, \frac{c+d}{2}) dx + \frac{1}{d-c} \int_c^d F(\frac{a+b}{2}, y) dy \right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x, y) dy dx \\ &\leq \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4}, \end{aligned} \quad (2.5)$$

whenever F is a convex function on $[a, b] \times [c, d] \subseteq \mathbb{R}^2$. Applying inequality (2.5) for the convex function $G(s, t) = ||| |A^{1-t} X B^s|^r ||| ||| |A^t X B^{1-s}|^r |||$ on $[0, 1] \times [0, 1]$ we get the following result.

Corollary 2.5. *Suppose that $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and $r \geq 0$. Then*

$$\begin{aligned} 2||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 &\leq \frac{1}{1-2\alpha} \int_{\alpha}^{1-\alpha} ||| |A^s X B^{\frac{1}{2}}|^r ||| ||| |A^{1-s} X B^{\frac{1}{2}}|^r ||| ds \\ &\quad + \frac{1}{1-2\beta} \int_{\beta}^{1-\beta} ||| |A^{\frac{1}{2}} X B^{1-t}|^r ||| ||| |A^{\frac{1}{2}} X B^t|^r ||| dt \\ &\leq \frac{2}{(1-2\alpha)(1-2\beta)} \int_{\alpha}^{1-\alpha} \int_{\beta}^{1-\beta} ||| |A^s X B^{1-t}|^r ||| ||| |A^{1-s} X B^t|^r ||| dt ds \\ &\leq ||| |A^{\alpha} X B^{1-\beta}|^r ||| ||| |A^{1-\alpha} X B^{\beta}|^r ||| + ||| |A^{1-\alpha} X B^{1-\beta}|^r ||| ||| |A^{\alpha} X B^{\beta}|^r ||| \end{aligned}$$

for all $\alpha, \beta \in [0, \frac{1}{2})$ and

$$\begin{aligned} 2||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 &\leq \frac{1}{2\alpha-1} \int_{1-\alpha}^{\alpha} ||| |A^s X B^{\frac{1}{2}}|^r ||| ||| |A^{1-s} X B^{\frac{1}{2}}|^r ||| ds \\ &\quad + \frac{1}{2\beta-1} \int_{1-\beta}^{\beta} ||| |A^{\frac{1}{2}} X B^{1-t}|^r ||| ||| |A^{\frac{1}{2}} X B^t|^r ||| dt \\ &\leq \frac{2}{(2\alpha-1)(2\beta-1)} \int_{1-\alpha}^{\alpha} \int_{1-\beta}^{\beta} ||| |A^s X B^{1-t}|^r ||| ||| |A^{1-s} X B^t|^r ||| dt ds \\ &\leq ||| |A^{\alpha} X B^{1-\beta}|^r ||| ||| |A^{1-\alpha} X B^{\beta}|^r ||| + ||| |A^{1-\alpha} X B^{1-\beta}|^r ||| ||| |A^{\alpha} X B^{\beta}|^r ||| \end{aligned}$$

for all $\alpha, \beta \in (\frac{1}{2}, 1]$.

Proof. Let $G(s, t) = ||| |A^t X B^{1-t}|^r ||| ||| |A^{1-t} X B^t|^r |||$. If we replace a by α , b by $1 - \alpha$, c by β and d by $1 - \beta$ ($\alpha, \beta \in [0, \frac{1}{2})$) for the convex function G in (2.5) we reach the first inequality and if we replace a by $1 - \alpha$, b by α , c by $1 - \beta$ and d by β ($\alpha, \beta \in (\frac{1}{2}, 1]$) in (2.5) we obtain the second inequality. \square

The spacial case $\alpha = \beta = 1$ of Theorem 2.5 reads as follows.

Corollary 2.6. *Suppose that $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$ and $r \geq 0$. Then*

$$\begin{aligned} 2 ||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 &\leq \int_0^1 ||| |A^s X B^{\frac{1}{2}}|^r ||| ||| |A^{1-s} X B^{\frac{1}{2}}|^r ||| ds \\ &\quad + \int_0^1 ||| |A^{\frac{1}{2}} X B^{1-t}|^r ||| ||| |A^{\frac{1}{2}} X B^t|^r ||| dt \\ &\leq 2 \int_0^1 \int_0^1 ||| |A^s X B^{1-t}|^r ||| ||| |A^{1-s} X B^t|^r ||| dt ds \\ &\leq ||| |AX|^r ||| ||| |XB|^r ||| + ||| |X|^r ||| ||| |AXB|^r |||. \end{aligned}$$

3. FURTHER REFINEMENTS OF THE CAUCHY-SCHWARZ INEQUALITY

In this section, we establish some refinements of the Cauchy-Schwarz inequality. The following result, derived in the recent papers [8, 9].

Lemma 3.1. [8] *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function and $\delta \in [a, b]$, $p \in (0, 1)$ be fixed parameters. Then the function $\varphi : [a, b] \rightarrow \mathbb{R}$, defined by*

$$\varphi(t) = (1 - p)f(\delta) + pf(t) - f((1 - p)\delta + pt)$$

is decreasing on $[a, \delta]$ and is increasing on $[\delta, b]$.

In the next result, we show a refinement of the right side of inequality (1.2).

Theorem 3.2. *Let $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$, $r \geq 0$, $\mu \in [0, 1]$, $p \in (0, 1)$ and let $||| \cdot |||$ be any unitarily invariant norm. Then*

$$\begin{aligned} &||| |AX|^r ||| ||| |XB|^r ||| - ||| |A^\mu X B^{1-\mu}|^r ||| ||| |A^{1-\mu} X B^\mu|^r ||| \\ &\geq \frac{1}{p} \left(f\left(\frac{1-p}{2}\right) - f\left(\frac{1-p}{2} + p\mu\right) \right) \geq 0, \end{aligned} \quad (3.1)$$

where $f(t) = ||| |A^t X B^{1-t}|^r ||| ||| |A^{1-t} X B^t|^r |||$ ($t \in [0, 1]$).

Proof. Assume that the functions $f(t) = ||| |A^t X B^{1-t}|^r ||| ||| |A^{1-t} X B^t|^r |||$ ($t \in [0, 1]$) and $\varphi(\mu) = (1 - p)f\left(\frac{1}{2}\right) + pf(\mu) - f\left(\frac{1-p}{2} + p\mu\right)$ ($\mu \in [0, 1]$). Using Lemma 3.1, we see

that φ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$. Let that $\mu \in [0, \frac{1}{2}]$. Since φ is decreasing on $[0, \frac{1}{2}]$, we have $\varphi(0) \geq \varphi(\mu)$, that is,

$$pf(0) - f\left(\frac{1-p}{2}\right) \geq pf(\mu) - f\left(\frac{1-p}{2} + p\mu\right),$$

whence

$$f(0) - f(\mu) \geq \frac{1}{p} \left[f\left(\frac{1-p}{2}\right) - f\left(\frac{1-p}{2} + p\mu\right) \right], \quad (3.2)$$

which yields desired inequality. Note, the right hand side of (3.2) is decreasing and $\frac{1-p}{2} + p\mu \geq \frac{1-p}{2}$. Now let $\mu \in [\frac{1}{2}, 1]$. So $0 \leq 1 - \mu \leq \frac{1}{2}$. By the symmetry property of (3.2) with respect to μ , if we replace μ by $1 - \mu$, then

$$f(0) - f(1 - \mu) \geq \frac{1}{p} \left[f\left(\frac{1-p}{2}\right) - f\left(\frac{1-p}{2} - p\mu\right) \right],$$

which is reduce to (3.1) since $f(1 - \mu) = f(\mu)$, ($\mu \in [0, 1]$). \square

By the same strategy as in the proof of Theorem 3.3, we get a refinement of the left side inequality (1.2).

Theorem 3.3. *Let $A, B \in \mathcal{S}_n$, $X \in \mathcal{M}_n$, $r \geq 0$, $\mu \in [0, 1]$, $p \in (0, 1)$ and let $||| \cdot |||$ be any unitarily invariant norm. Then*

$$\begin{aligned} & ||| |A^\mu X B^{1-\mu}|^r ||| \quad ||| |A^{1-\mu} X B^\mu|^r ||| - ||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 \\ & \geq \frac{1}{p} \left(f\left(\frac{1-p}{2} + p\mu\right) - ||| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r |||^2 \right) \geq 0, \end{aligned}$$

where $f(t) = ||| |A^t X B^{1-t}|^r ||| \quad ||| |A^{1-t} X B^t|^r |||$ ($t \in [0, 1]$).

4. SOME INEQUALITIES INVOLVING NUMERICAL RADIUS

In this section we show inequalities involving Heinz type numerical radius. A continuous real valued function f defined on an interval (a, b) with $a \geq 0$ is called Kwong function if the matrix

$$\left(\frac{f(a_i) + f(a_j)}{a_i + a_j} \right)_{i,j=1}^n$$

is positive semidefinite for any distinct real numbers a_1, \dots, a_n in (a, b) .

Lemma 4.1. [1, Corollary 4] *Let $A = [a_{ij}] \in \mathcal{M}_n$ be positive semidefinite. Then*

$$\|S_A\|_\omega = \max_i a_{ii}.$$

Lemma 4.2. [14, Theorem 3.4] (*Spectral Decomposition*) Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then A is normal if and only if there exists a unitary matrix U such that

$$U^*AU = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

In particular, A is positive definite if and only if the λ_j ($1 \leq j \leq n$) are positive.

Theorem 4.3. Suppose that $A \in \mathcal{P}_n$, $X \in \mathcal{M}_n$, $\alpha \in [0, 1]$ and $\frac{f}{g}$ be a Kwong function such that $f(t)g(t) \leq t$ ($t \geq 0$). Then

$$\omega(f(A)Xg(A) + g(A)Xf(A)) \leq \omega(AX + XA)$$

Proof. Applying Lemma 4.2, we can assume that $A = \text{diag}(a_1, a_2, \dots, a_n)$ is diagonalize, where a_j ($j = 1, 2, \dots, n$) are positive numbers. Let $Z = [z_{ij}] \in \mathcal{M}_n$ with the entries $z_{ij} = \frac{f(a_i)g(a_i) + f(a_j)g(a_j)}{a_i + a_j}$ ($1 \leq i, j \leq n$). Since $\frac{f}{g}$ is a Kwong function,

$$Z = S \left(\frac{f(a_i)g^{-1}(a_i) + f(a_j)g^{-1}(a_j)}{a_i + a_j} \right)_{i,j=1}^n S$$

is positive semidefinite where $S = \text{diag}(g(a_1), \dots, g(a_n))$. It follows from Lemma 4.1 that

$$\|S_Z\|_\omega = \max_i z_{ii} = \frac{f(a_i)g(a_i)}{a_i} \leq 1,$$

or equivalently, $\frac{\omega(Z \circ X)}{\omega(X)} \leq 1$ ($0 \neq X \in \mathcal{M}_n$). Let $E = [\frac{1}{a_i + a_j}]$ and $D = [f(a_i)g(a_i) + f(a_j)g(a_j)] \in \mathcal{M}_n$. Hence

$$\omega(D \circ E \circ X) = \omega(Z \circ X) \leq \omega(X) \quad (X \in \mathcal{M}_n).$$

Let the matrix C be the entrywise inverse of E , i.e., $C \circ E = J$. Thus $\omega(D \circ X) \leq \omega(C \circ X)$ ($X \in \mathcal{M}_n$). Hence

$$\omega(f(A)Xg(A) + g(A)Xf(A)) \leq \omega(AX + XA).$$

□

Using $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in Theorem 4.3 we get the following Heinz type inequality in the following result.

Corollary 4.4. Suppose that $A \in \mathcal{P}_n$, $X \in \mathcal{M}_n$ and $\alpha \in [0, 1]$. Then

$$\omega(A^\alpha X A^{1-\alpha} + A^{1-\alpha} X A^\alpha) \leq \omega(AX + XA)$$

Kwong [10] showed that the set Kwong functions on $(0, \infty)$ includes all non-negative operator monotone functions f on $(0, \infty)$.

Example 4.5. The function $f(t) = \log(t + 1)$ is operator monotone on the interval $(0, \infty)$ [13]. If $g(t) = \frac{t}{f(t)}$, then, by Theorem 4.3, for every unitarily invariant norm $||| \cdot |||$, $A \in \mathcal{P}_n$ and $X \in \mathcal{M}_n$ we have

$$\omega(\log(A + 1)XA\log(A + 1)^{-1} + A\log(A + 1)^{-1}X\log(A + 1)) \leq \omega(AX + XA).$$

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